Function notation

Given $f(x)=\dfrac{3x-2}{2x+1}$ and $g(x)=\dfrac{2x-1}{3x-2}$

$f(0)=-2$

$f\left(\tfrac{2}{3}\right)=0$

$f(-1)=5$

$f(2)=\dfrac{4}{5}$

$$f(2x)=\frac{3(2x)-2}{2(2x)+1}=\frac{6x-2}{4x+1}$$

$$f(x+3)=\frac{3(x+3)-2}{2(x+3)+1}=\frac{3x+3-2}{2x+6+1}=\frac{3x+7}{2x+7}$$

$$f(g(x))=\frac{3\left(\dfrac{2x-1}{3x-2}\right)-2}{2\left(\dfrac{2x-1}{3x-2}\right)+1}=\frac{\dfrac{6x-3}{3x-2}-2}{\dfrac{4x-2}{3x-2}+1}=\frac{\dfrac{6x-3-2(3x-2)}{3x-2}}{\dfrac{4x-2+3x-2}{3x-2}}=\frac{1}{7x-4}$$

$$g(f(x))=\frac{4x-5}{5x-8}$$

$$f(x)\cdot g(x)=\frac{2x-1}{2x+1}$$

$$f(x)+g(x)=\frac{13x^2-12x+3}{(2x+1)(3x-2)}$$

$$\frac{f(x)}{g(x)}=\frac{9x^2-12x+4}{(2x+1)(2x-1)}$$

Given $f(x) = \frac{3x-2}{2x+1}$ $g(x) = \frac{2x-1}{3x-2}$

find $f(0) = -2$

$f(\frac{2}{3}) = 0$

$f(-1) = 5$

$(2) = \frac{4}{5}$

$f(2x) = \frac{6x-2}{4x+1}$

$f(x+3) = \frac{3x+9-2}{2x+6+1} = \frac{3x+7}{2x+7}$

$fg = \frac{2x-1}{2x+1}$

$f+g = \frac{9x^2-12x+4+4x^2-1}{(\quad)(\quad)}$

$\frac{13x^2-12x+3}{(2x+1)(3x-2)}$

$$f(g(x)) = f\left(\frac{2x-1}{3x-2}\right) = \frac{3\left(\frac{2x-1}{3x-2}\right)-2}{2\left(\frac{2x-1}{3x-2}\right)+1}$$

$$= \frac{6x-3-6x+4}{4x-2+3x-2}$$

$$= \frac{1}{7x-4}$$

$$g(f(x)) = g\left(\frac{3x-2}{2x+1}\right) = \frac{2\left(\frac{3x-2}{2x+1}\right)-1}{3\left(\frac{3x-2}{2x+1}\right)-2}$$

$$\frac{6x-4-2x-1}{9x-6-4x-2}$$

$$\frac{4x-5}{5x-8}$$

$$\frac{f}{g} = \frac{\frac{3x-2}{2x+1}}{\frac{2x-1}{3x-2}}$$

$$= \frac{9x^2-12x+4}{(2x+1)(2x-1)}$$

$$f\left(\frac{x-2}{2x-3}\right) = \frac{2\left(\frac{x-2}{2x-3}\right)+3}{\frac{x-2}{2x-3}-2}$$

$$\frac{2x-4+6x-9}{x-2-4x+6}$$

$$\frac{8x-13}{-3x+4}$$

$$g(f) = g\left(\frac{2x+3}{x-2}\right) = \frac{\frac{2x+3}{x-2}-2}{2\left(\frac{2x+3}{x-2}\right)-3}$$

$$= \frac{2x+3-2x+4}{4x+6-3x+6}$$

$$\frac{7}{x+12}$$

$$f^{-1}(x) = \qquad f(x) = \frac{2x+3}{x-2}$$

$$x = \frac{2y+3}{y-2}$$

$$xy-2x = 2y+3$$

$$y(x-2) = 2y+3$$

$$y = \frac{2x+3}{x-2}$$

Section 6.5 Work Problems.

Rate of work is written (for example) as "3 jobs per day" or "5 pages per hour".
Writing these in an algebraic form gives: $3\frac{\text{jobs}}{\text{day}}$, $5\frac{\text{pages}}{\text{hour}}$.

If you multiply $5\frac{\text{pages}}{\text{hour}}$ times 4 hours, the hours reduce and the result is 20 pages. This is exactly as you would expect.

Example:

Sue can mow the lawn in 4 hours and Lenny can do the job in 6 hours. How long will it take them if they work together?

Sue: 1 job --- 4 hours therefore Sue's rate is $\frac{1}{4}$. That is she can do one-fourth of the job per hour.

Likewise, Lenny's rate is $\frac{1}{6}$.

Let t = hours working together.

$$\frac{1}{4}t + \frac{1}{6}t = 1$$

$$12\left(\frac{1}{4}t + \frac{1}{6}t = 1\right)$$

$$3t + 2t = 12$$

$$5t = 12$$

$$t = \frac{12}{5}$$

$$t = 2\frac{2}{5}$$

$t = 2$ hours 24 minutes

Sue and Lenny will complete the job in 2 hours and 24 minutes.

Example 2, page 391

It takes Manuel 9 hours longer than Zoe to rebuild an engine. Working together, they can do the job in 20 hours. How long would it take each, working alone, to rebuild an engine?

We have two players in this problem. Each needs an entry in the Let statement.
Who works the fastest? Who takes the least amount of time?

Zoe. We will add 9 hours to Zoe's hours to get Manuel's hours.

Let Z = hours needed by Zoe
Z + 9 = hours needed by Manuel

What is Zoe's rate of work? $\frac{1}{Z}$

What is Manuel's rate of work? $\frac{1}{Z+9}$

Translate the story into algebra.

$$\frac{20}{Z}+\frac{20}{Z+9}=1$$
$$Z(Z+9)\left(\frac{20}{Z}+\frac{20}{Z+9}=1\right)$$
$$20Z+20(Z+9)=Z(Z+9)$$
$$20Z+20Z+180=Z^2+9Z$$
$$0=Z^2-31Z-180$$
$$0=(Z-36)(Z+5)$$

Z = 36 means that Zoe would need 36 hours to do the job by herself. The Z = – 5 is meaningless and is rejected because it is negative.

Zoe needs 36 hours to rebuild the motor. Manuel needs 45 hours to rebuild the motor.

Example 3 pg 392. I do not care for the set up of example 3 in the text.

My preferred set up follows:
The slow bike is the mountain bike so we let m = speed of mountain bike. The racing bike is 15 mph faster so its rate is m + 15. The 80 and 50 came from the story.

	R	T	D
Road Bike	$m+15$		80
Mtn Bike	m		50

We fill in the missing items in the table using the fact that R * T = D

	R	T	D
Racer	$m+15$	$\frac{80}{m+15}$	80
Mtn Bike	m	$\frac{50}{m}$	50

The story says the times are the same so $\frac{80}{m+15}=\frac{50}{m}$. Solve.

Example 4 pg 393 again is not set up as I like it.

We use:

Let c = rate of the current
Up stream is slowed by the current thus 10 – c.

	R	T	D
Up	$10-c$		24
Down	$10+c$		24

As before, we fill in the table using the R * T = D

	R	T	D
Up	$10-c$	$\frac{24}{10-c}$	24
Down	$10+c$	$\frac{24}{10+c}$	24

The Up stream time plus the down stream time gives total time of 5 hours.

$$\frac{24}{10-c}+\frac{24}{10+c}=5$$
etc.

Example:

The small air conditioner can clean the air in a 12 ft by 14 ft room in 10 minutes. The larger air conditioner can clean the same size room in only 6 minutes. How long would it take both machines working together in such a room?

Let t = minutes working together

$$\frac{t}{6}+\frac{t}{10}=1$$

$$30\left(\frac{t}{6}+\frac{t}{10}=1\right)$$

$$5t+3t=30$$

$$8t=30$$

$$t=\frac{15}{4}$$

$$\frac{15}{4}=3\frac{3}{4}=3\frac{45}{60}$$

The machines working together require 3 minutes and 45 seconds to clean the room.

Problem 14 page 397
We have 2 machines. One is faster than the other. The faster machine requires fewer hours.

We let f = hours needed by faster machine

This means that f hours are required for 1 Job. We might say

f hrs = 1 job so dividing by f we have $1\text{hr} = \frac{1}{f}$ job.

That is the RATE per hour of work.

Consider, 8 hours to do a job. In one hour, $\frac{1}{8}$ of the job would be done. RATE.

Continuing with our set up we have:

$2f$ = hours required by slower machine.

$$\frac{15}{f}+\frac{15}{2f}=1$$

$$2F\left(\frac{15}{f}+\frac{15}{2F}=1\right)$$

$$30+15=2f$$

$$45=2f$$

$$\frac{45}{2}=f$$

The faster machine requires 22.5 hours to print the manuals and the slower machine could do the job in 45 hours.

Section 6.5 Problems starting page 397

A community water tank can be filled in 18 hous by the town office well alone and in 22 hr by the high school well alone. How long will it take to fill the tank if both wells are working?

We have two wells. A faster well and a slower well. The question asks for “The time working together” so that is our let statement

Let t = time both wells work together.

The faster well requires 18 hours to do the job by itself therefore its rate per hour is $\frac{1}{18}$.

Likewise, the slower well’s rate is $\frac{1}{22}$.

Remember: rate = $\frac{\text{work}}{\text{hour}}$ and $\frac{\text{work}}{\cancel{\text{hour}}} \cdot \cancel{\text{hour}} = \text{work}$.

We set up our translation of the word problem: $\frac{t}{18}+\frac{t}{22}=1$ that is "one job done".

The final answer is: Working together both wells can fill the tank in 9 hours and 54 minutes.

The HP Officejet takes twice the time required by the Canon Imageclass to photocopy brochures for the Arts Council concert. If working together the two machines can complete the job in24 minutes, how long would it take each machine working alone to copy the brochures?

We are given the time together. We are asked for the time required by each machine when they work independently. This gives us our Let statement:

Let f = minutes needed by faster machine. (f for faster)
2f = minutes needed by slower machine. – it takes twice as long ie 2 times f

The rates for each are the reciprocals. We multiply the time working together times the rates to get the work done by each machine:

$$\frac{24}{f}+\frac{24}{2f}=1$$

26) Moving sidewalks

The moving sidewalk moves 1.8 ft/sec. Walking on the moving sidewalk, Camille travels 105 ft forward in the time it takes to travel 51 ft in the opposite direction. How fast would Camille be walking on a nonmoving sidewalk?

This is a rate times time = distance problem. This problem involves the concept similar to the boat in a current and the plane and wind speed. The current/wind/sidewalk assists the person when going "with" the assist and impedes the person when going in the opposite direction. The question asks for Camille's walking speed.

Let C = Camille's walking speed.

	R	T	=	D
with	$C+\frac{9}{5}$		=	105
agnst	$C-\frac{9}{5}$		=	51

We need our rates to be single fractions:

	R	T	=	D
with	$\frac{5C+9}{5}$		=	105
agnst	$\frac{5C-9}{5}$		=	51

The Distance divided by the rate gives the time thus:

	R	T	=	D
with	$\frac{5C+9}{5}$	$\frac{5\cdot 105}{5C+9}$	=	105
agnst	$\frac{5C-9}{5}$	$\frac{5\cdot 51}{5C-9}$	=	51

Since the times are the same, we equate them and solve.

30) Another RT = D problem

	R	T	=	D
A train	A		=	230
B train	$A+12$		=	290

	R	T	=	D
A train	A	$\frac{230}{A}$	=	230
B train	$A+12$	$\frac{290}{A+12}$	=	290

$\frac{230}{A}=\frac{290}{A+12}$ Multiply by the LCD $A(A+12)$ and solve.

50) This problem's job is to *empty* a full tub.

Justine's hot tub can be filled in 10 min and drained in 8 minutes. How long will it take to empty a full tub if the water is left on?

Let t = time working together

$\frac{t}{8}-\frac{t}{10}=1$ Notice the negative sign. The drain is impeding job; not helping.

Section 6.6 Division and 6.7 synthetic division / synthetic substitution

The key phrase used in division is:

"What times this gives that *exactly*?"

Example 1

$\dfrac{-25x^3 + 20x^2 - 3x + 7}{5x}$ becomes: $5x\overline{)-25x^3 + 20x^2 - 3x + 7}$

and we ask the quote….. What times 5x gives $--25x^3$ *Exactly*? $-5x^2$. Put that in the proper column and move to the next term and repeat the quote until all terms are accounted for. Final answer for this example is: $-5x^2 + 4x - \dfrac{3}{5} + \dfrac{7}{5x}$.

Example 2

$$2x - 1\overline{)4x^4 - 8x^3 - 5x^2 + 14x + 5}$$

Look ONLY at the 2x and ask the quote: "What times 2x gives $4x^4$ exactly?" Put the $2x^3$ in the "cube column". Multiply the $2x^3$ times $2x - 1$ giving $4x^4 - 2x^3$ placing them in the proper column.

		$2x^3$			
$2x-1$	$4x^4$	$-8x^3$	$-5x^2$	$+14x$	$+5$
	$4x^4$	$-2x^3$			

We need to "subtract". To do this with the least chance of error we "change the sign and add". Thus the problem looks like this just before we add:

		$2x^3$			
$2x-1$	$4x^4$	$-8x^3$	$-5x^2$	$+14x$	$+5$
	$\mp 4x^4$	$\pm 2x^3$			

Now we add to get $-6x^3$

$$\begin{array}{r|rrrrr} & & 2x^3 & & & \\ \hline 2x-1 & 4x^4 & -8x^3 & -5x^2 & +14x & +5 \\ & \mp 4x^4 & \pm 2x^3 & & & \\ \hline & & -6x^3 & & & \end{array}$$

We being the process again….

What times 2x gives $-6x^3$ exactly?

$$\begin{array}{r|rrrrr} & & 2x^3 & -3x^2 & & \\ \hline 2x-1 & 4x^4 & -8x^3 & -5x^2 & +14x & +5 \\ & \mp 4x^4 & \pm 2x^3 & & & \\ \hline & & -6x^3 & & & \\ & & -6x^3 & +3x^2 & & \\ \hline \end{array}$$

We change the signs and add to get:

$$\begin{array}{r|rrrrr} & & 2x^3 & -3x^2 & & \\ \hline 2x-1 & 4x^4 & -8x^3 & -5x^2 & +14x & +5 \\ & \mp 4x^4 & \pm 2x^3 & & & \\ \hline & & -6x^3 & & & \\ & & \pm 6x^3 & \mp 3x^2 & & \\ \hline & & & -8x^2 & & \end{array}$$

Notice that we change the sign ABOVE the original sign NOT on top of the original sign. For example, we do NOT draw a vertical line through the minus sign making it a plus sign!

The steps are repeated until we have a accounted for the terms in the dividend.

This example's conclusion will look like this:

		$2x^3$	$-3x^2$	$-4x$	$+5$
$2x-1$	$4x^4$	$-8x^3$	$-5x^2$	$+14x$	$+5$
	$\mp 4x^4$	$\pm 2x^3$			
		$-6x^3$			
		$\pm 6x^3$	$\mp 3x^2$		
			$-8x^2$		
			$\pm 8x^2$	$\mp 4x$	
				$10x$	
				$\mp 10x$	-5
					10

The quotient is: $2x^3 - 3x^2 - 4x + 5 + \dfrac{10}{2x-1}$

Written in $f(x) = D(x)Q(x) + R(x)$ Where
$f(x)$ is the original expression
$D(x)$ is the divisor
$Q(x)$ is the quotient
$R(x)$ is the remainder

$$4x^4 - 8x^3 - 5x^2 + 14x + 5 = (2x-1)\left(2x^3 - 3x^2 - 4x + 5\right) + 10$$

There is a lot of duplication of effort in this style of division. One method to reduce the duplication is called "Synthetic Division"

The form of the Remainder Theorem is: $P(x) = (x-a)Q(x) + R(a)$

The first item to notice is the divisor is $(x-a)$. The coefficient of x is one! Our divisor was $2x - 1$. This becomes: $2\left(x - \frac{1}{2}\right)$. The factor, two, will show up as common factor in the quotient found using synthetic division.

	4	–8	–5	14	5
$\frac{1}{2}$	4	–6	–8	10	10

The top row is composed of the coefficients of the original expression. The one-half is from the $\left(x - \frac{1}{2}\right)$.

The "rule" for synthetic division is ….

1. Bring down the first number.
2. Multiply the outside number times the "brought down" number and then add to the next number. Thus one-half times 4 plus – 8 gives – 6.
3. Multiply the outside number times the – 6 plus – 5 gives – 8.
4. The process is repeated until the last item has been processed.

Reading across the top, notice the 4, – 6, – 8, 10 are the coefficients of the quotient we found before.

We now have

$$4x^4 - 8x^3 - 5x^2 + 14x + 5 = \left(x - \frac{1}{2}\right)\left(4x^3 - 6x^2 - 8x + 10\right) + 10$$

Notice the quotient has a common factor of 2.

$$4x^4 - 8x^3 - 5x^2 + 14x + 5 = \left(x - \frac{1}{2}\right)2\left(2x^3 - 3x^2 - 4x + 5\right) + 10$$

This finally becomes

$4x^4 - 8x^3 - 5x^2 + 14x + 5 = (2x - 1)\left(2x^3 - 3x^2 - 4x + 5\right) + 10$ as we had before.

There is an interesting alternative interpretation for synthetic division. It is called synthetic substitution.

Given: $(x+3)\overline{)2x^3 + 7x^2 - 5}$ Note: $(x-(-3))\overline{)2x^3 + 7x^2 - 5}$

setting up the synthetic division….

	2	7	0	–5
–3	2	1	–3	4

This gives us $\left(x - (-3)\right)\left(2x^2 + x - 3\right) + 4$

Consider this….

	2	7	0	–5
–3	2	1	–3	4
–2	2	3	–6	7
–4	2	–1	4	–21

Each line is computed as if the previous lines were not there.

Now, given: $f(x) = 2x^3 + 7x^2 - 5$, find $f(-3), f(-4), f(-5)$.

Why does this happen? Remember we had $P(x) = (x-a)Q(x) + R(a)$.

The $R(a)$ is the remainder after division by a and it is also the function value at x = a because when (x = a) the term $(x-a)Q(x)$ is zero and the left over is the function value!

We use the same procedure to apply synthetic division as we do to apply synthetic substitution.